Multiple integration 2

Frames 1 to 77

Learning outcomes

When you have completed this Programme you will be able to:

- Evaluate double integrals and surface integrals
- Relate three-dimensional Cartesian coordinates to cylindrical and spherical polar forms
- Evaluate volume integrals in Cartesian coordinates and in cylindrical and spherical polar coordinates
- Use the Jacobian to convert integrals given in Cartesian coordinates into general curvilinear coordinates in two and three dimensions

Double integrals

1

Let us start off with an example with which we are already familiar.

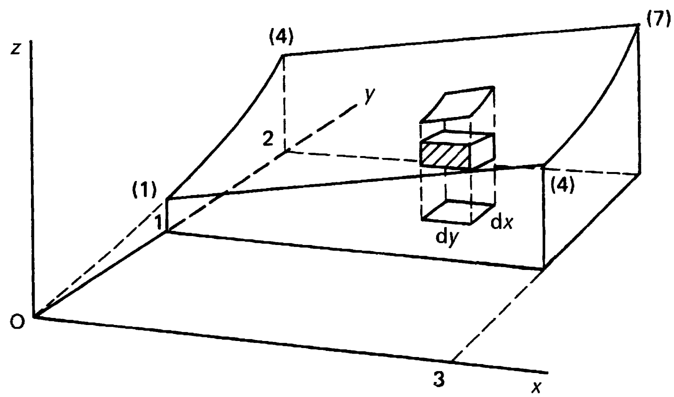
Example 1

A solid is enclosed by the planes $z = 0$, $y = 1$, $y = 2$, $x = 0$, $x = 3$ and the surface $z = x + y^2$. We have to determine the volume of the solid so formed.

First take some care in sketching the figure, which is

....

2



In the plane $y = 1$, $z = x + 1$, i.e. a straight line joining (0, 1, 1) and (3, 1, 4)

In the plane $y = 2$, $z = x + 4$, i.e. a straight line joining (0, 2, 4) and (3, 2, 7)

In the plane $x = 0$, $z = y^2$, i.e. a parabola joining (0, 1, 1) and (0, 2, 4)

In the plane $x = 3$, $z = 3 + y^2$, i.e. a parabola joining (3, 1, 4) and (3, 2, 7).

Consideration like this helps us to visualise the problem and the time involved is well spent.

Now we can proceed.

The element of volume $\delta v = \delta x \, \delta y \, \delta z$

Then the total volume $V = \iiint dx\, dy\, dz$ between appropriate limits in each case.

▶

We could also have said that the element of area on the $z = 0$ plane

$\delta a = \delta y\,\delta x$

and that the volume of the column $\quad \delta v_c = z\,\delta a = z\,\delta x\,\delta y$

Then, since $z = x + y^2$, this becomes $\quad \delta v_c = (x + y^2)\,\delta x\,\delta y$

Summing in the usual way then gives

$$\begin{aligned} V &= \int z\,\mathrm{d}a \\ &= \int_R\!\int (x + y^2)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

where R is the region bounded in the x–y plane.

Now we insert the appropriate limits and complete the integration

$V = \ldots\ldots\ldots\ldots$

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$$\boxed{V = 11{\cdot}5 \text{ cubic units}}$$

Because

$$\begin{aligned} V &= \int_{y=1}^{y=2}\int_{x=0}^{x=3}(x + y^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_1^2 \left[\frac{x^2}{2} + xy^2\right]_{x=0}^{x=3}\mathrm{d}y \\ &= \int_1^2 \left(\frac{9}{2} + 3y^2\right)\mathrm{d}y \\ &= \left[\frac{9}{2}y + y^3\right]_1^2 \\ &= 11{\cdot}5 \end{aligned}$$

$\therefore\ V = 11{\cdot}5$ cubic units

Although we have found a volume, this is, in fact, an example of a *double integral* since the expression for z was a function of position in the x–y plane within the closed region

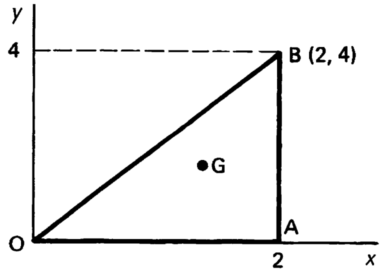
$$\text{i.e.}\quad \begin{aligned} I &= \int_R\!\int f(x, y)\,\mathrm{d}a \\ &= \int_R\!\int f(x, y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

In this particular case, R is the region in the x–y plane bounded by $x = 0$, $x = 3$, $y = 1$, $y = 2$.

▶

Example 2

A triangular thin plate has the dimensions shown and a variable density ρ where $\rho = 1 + x + xy$.

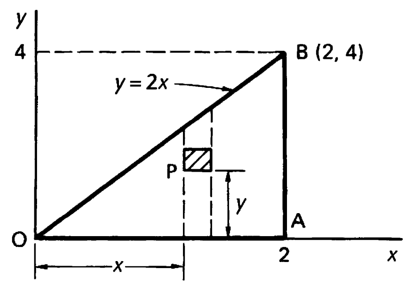


We have to determine

(a) the mass of the plate

(b) the position of its centre of gravity G.

(a) Consider an element of area at the point P (x, y) in the plate



$$\delta a = \delta x\, \delta y$$

The mass δm of the element is then

$$\delta m = \rho\, \delta x\, \delta y$$

$$\therefore \text{ Total mass } M = \int_R\int \mathrm{d}m = \int_R\int \rho\, \mathrm{d}x\, \mathrm{d}y$$

Now we insert the limits and complete the integration, remembering that $\rho = (1 + x + xy)$

$M = \dots\dots\dots\dots$

4

$$\boxed{M = 17\frac{1}{3}}$$

Because we have

$$M = \int_R\int \rho\, \mathrm{d}x\, \mathrm{d}y = \int_{x=0}^{x=2}\int_{y=0}^{y=2x} (1 + x + xy)\, \mathrm{d}y\, \mathrm{d}x$$

$$= \int_0^2 \left[y + xy + \frac{xy^2}{2}\right]_{y=0}^{y=2x} \mathrm{d}x$$

$$= \int_0^2 \{2x + 2x^2 + 2x^3\}\mathrm{d}x$$

$$= \left[x^2 + \frac{2x^3}{3} + \frac{x^4}{2}\right]_0^2 = 17\frac{1}{3}$$

(b) To find the position of the centre of gravity, we need to know

$\dots\dots\dots\dots$

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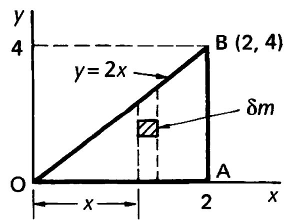
$$\boxed{\text{the sum of the moments of mass about OY and OX}}$$

(1) To find $\bar{x}$, we take moments about OY.



Moment of mass of element about OY

$$= x\,\delta m$$
$$= x(1 + x + xy)\,\delta x\,\delta y$$

$$\therefore \text{ Sum of first moments} = \int_R\int (x + x^2 + x^2y)\,dx\,dy$$
$$= \dots\dots\dots\dots$$

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$$\boxed{26\frac{2}{15}}$$

$$\text{Because sum of first moments} = \int_{x=0}^{x=2}\int_{y=0}^{y=2x}(x + x^2 + x^2y)\,dy\,dx$$
$$= \int_0^2\left[xy + x^2y + \frac{x^2y^2}{2}\right]_{y=0}^{y=2x}dx$$
$$= \int_0^2\{2x^2 + 2x^3 + 2x^4\}dx$$
$$= 2\int_0^2(x^2 + x^3 + x^4)\,dx$$
$$= 2\left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right]_0^2 = 26\frac{2}{15}$$

Now $M\bar{x} = \text{sum of moments} \quad \therefore \; \bar{x} = \dots\dots\dots\dots$

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$$\boxed{\bar{x} = 1{\cdot}508}$$

We found previously that $M = 17\frac{1}{3} \quad \therefore \; \left(17\frac{1}{3}\right)\bar{x} = 26\frac{2}{15}$

which gives $\bar{x} = 1\frac{33}{65} = 1{\cdot}508$

(2) To find $\bar{y}$ we proceed in just the same way, this time taking moments about OX. Work right through it on your own.

$$\bar{y} = \dots\dots\dots\dots$$

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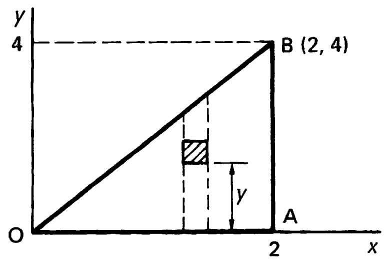
$$\boxed{\bar{y} = 1{\cdot}754}$$



Moment of element of mass δm about OX

$$= y\,\delta m = y(1 + x + xy)\,\delta x\,\delta y$$

$$\therefore \text{ Sum of first moments about OX} = \int_R\int (y + xy + xy^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2x} (y + xy + xy^2)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2 \left[\frac{y^2}{2} + \frac{xy^2}{2} + \frac{xy^3}{3}\right]_{y=0}^{y=2x} \mathrm{d}x$$

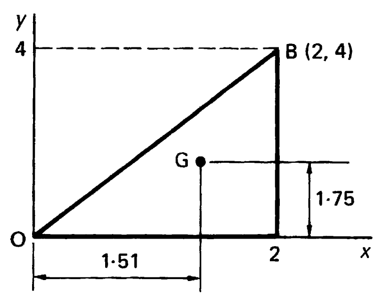
$$= \int_0^2 \left\{2x^2 + 2x^3 + \frac{8x^4}{3}\right\}\mathrm{d}x$$

$$= \left[\frac{2x^3}{3} + \frac{x^4}{2} + \frac{8x^5}{15}\right]_0^2$$

$$= 30\frac{2}{5}$$

$$\therefore\ M\bar{y} = 30\frac{2}{5} \quad \therefore\ \bar{y} = 30\frac{2}{5}\Big/17\frac{1}{3} = 1{\cdot}754$$

So we finally have:



Note that this again referred to a plane figure in the x–y plane.

Now let us move on to something slightly different

Surface integrals

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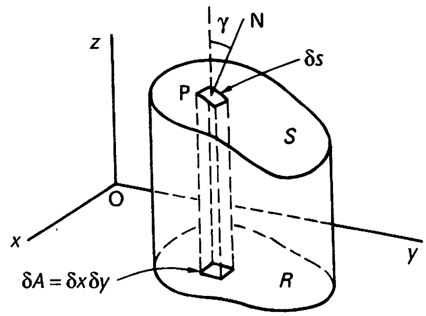
When the area over which we integrate is not restricted to the x–y plane, matters become rather more involved, but also more interesting.

If S is a two-sided surface in space and R is its projection on the x–y plane, then the equation of S is of the form $z = f(x, y)$ where f is a single-valued function and continuous throughout R.

Let δA denote an element of R and δS the corresponding element of area of S at the point $P(x, y, z)$ in S.



Let also $\phi(x, y, z)$ be a function of position on S (e.g. potential) and let γ denote the angle between the outward normal PN to the surface at P and the positive z-axis.

Then $\delta A \approx \delta S \cos\gamma$ i.e. $\delta S \approx \dfrac{\delta A}{\cos\gamma} = \delta A \sec\gamma$ and

$\sum \phi(x, y, z)\delta S$ is the total value of $\phi(x, y, z)$ taken over the surface S.

As $\delta S \to 0$, this sum becomes the integral

$$I = \int_S \phi(x, y, z)\,\mathrm{d}S$$

and, since $\delta S \approx \delta A \sec\gamma$, the result can be written

$$I = \int_R\int \phi(x, y, z)\sec\gamma\,\mathrm{d}x\,\mathrm{d}y \qquad \left(\gamma < \frac{\pi}{2}\right)$$

Notice that $\cos\gamma = \hat{\mathbf{n}} \cdot \mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction and $\hat{\mathbf{n}}$ is the unit normal to the surface at P.

With limits inserted for x and y, the integral seems straightforward, except for the factor $\sec\gamma$, which naturally varies over the surface S.

We can, in fact, show that $\sec\gamma = \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}$

(see Programme 17, Frames 69*ff*)

Therefore, the *surface integral* of $\phi(x, y, z)$ over the surface S is given by

(a) $$I = \int_S \phi(x, y, z)\,\mathrm{d}S \qquad (1)$$

or (b) $$I = \int_R\int \phi(x, y, z)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y \qquad (2)$$

where $z = f(x, y)$

▶

Note that, when $\phi(x, y, z) = 1$, then $I = \int_S \mathrm{d}S$ gives the area of the surface S.

$$\therefore \; S = \int_S \mathrm{d}S = \int_R \int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, \mathrm{d}x \, \mathrm{d}y \qquad (3)$$

Make a note of these three important results.

Then we will apply them to a few examples.

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Example 1

Find the area of the surface $z = \sqrt{x^2 + y^2}$ over the region bounded by $x^2 + y^2 = 1$.

$$S = \int_R \int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, \mathrm{d}x \, \mathrm{d}y$$

So we now find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ and determine $\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}$ which is ....

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$$\boxed{\sqrt{2}}$$

Because

$$z = (x^2 + y^2)^{1/2} \quad \therefore \; \frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2} 2y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\therefore \; 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2 + y^2}{x^2 + y^2} = 2$$

$$\therefore \; \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{2}$$

$$\therefore \; S = \sqrt{2} \int_R \int \mathrm{d}x \, \mathrm{d}y = \sqrt{2} \times \ldots\ldots\ldots\ldots$$

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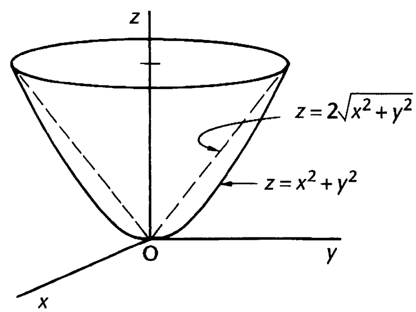
the area of the region R

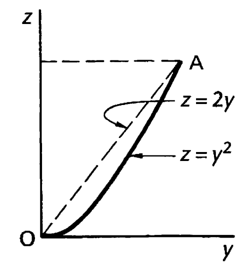
But R is bounded by $x^2 + y^2 = 1$, i.e. a circle, centre the origin and radius 1. $\therefore$ area $= \pi$

$$\therefore \; S = \sqrt{2}\int_R\int dx\,dy = \sqrt{2}\pi$$

Example 2

Find the area of the surface S of the paraboloid $z = x^2 + y^2$ cut off by the cone $z = 2\sqrt{x^2 + y^2}$.





We can find the point of intersection A by considering the y–z plane, i.e. put $x = 0$.

Coordinates of A are ....

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A (2, 4)

The projection of the surface S on the x–y plane is

....

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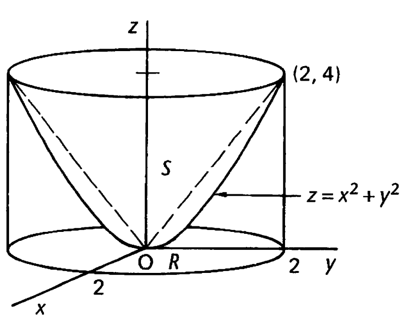
the circle $x^2 + y^2 = 4$



$$S = \int_R\int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

For this we use the equation of the surface S. The information from the projection R on the x–y plane will later provide the limits of the two stages of integration.

For the time being, then, $S =$ ....

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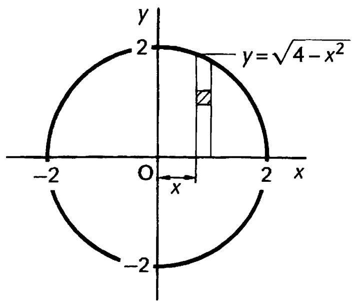
$$S = \int_R \int \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$



Using Cartesian coordinates, we could integrate with respect to y from $y = 0$ to $y = \sqrt{4 - x^2}$ and then with respect to x from $x = 0$ to $x = 2$. Finally, we should multiply by four to cover all four quadrants.

i.e. $$S = 4 \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

But how do we carry out the actual integration?

It becomes a lot easier if we use polar coordinates.

The same integral in polar coordinates is ....

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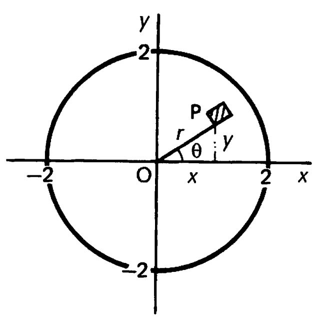
$$S = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$



$x = r\cos\theta; \qquad y = r\sin\theta$

$x^2 + y^2 = r^2 \qquad dx\, dy = r\, dr\, d\theta$

(refer to Frame 67)

$$S = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$\therefore\ S = $ ....

Finish it off.

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$$\boxed{S = 36{\cdot}18 \text{ square units}}$$

Because

$$S = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (1 + 4r^2)^{1/2} r \, dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{12} (1 + 4r^2)^{3/2} \right]_0^2 d\theta$$

$$= \frac{1}{12} \int_0^{2\pi} \{17^{3/2} - 1\} \, d\theta = 5{\cdot}7577 \Big[\theta \Big]_0^{2\pi} = 36{\cdot}18$$

Now on to Example 3.

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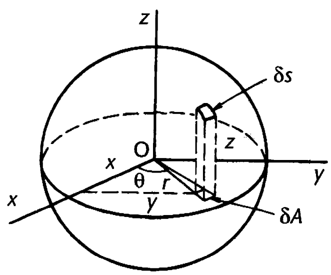
Example 3

To determine the moment of inertia of a thin spherical shell of radius a about a diameter as axis. The mass per unit area of shell is ρ.



Equation of sphere

$$x^2 + y^2 + z^2 = a^2$$

Mass of element $= m = \rho\,\delta S$

$$I \approx \Sigma m r^2 \approx \Sigma \rho\,\delta S r^2$$

Let us deal with the upper hemisphere

$$\therefore \; I_H = \int_S \rho r^2 \, dS$$

$$= \int_R \int \rho r^2 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx \, dy$$

Now determine the partial derivatives and simplify the integral as far as possible in Cartesian coordinates.

$$I_H = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_H = \int_R \int \rho r^2 \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dx \, dy}$$

In this particular example, R is, of course, the region bounded by the circle $x^2 + y^2 = a^2$ in the x–y plane.

Converting to polar coordinates

$$x = r\cos\theta; \quad y = r\sin\theta; \quad dx\,dy = r\,dr\,d\theta$$

the integral becomes $I_H = \ldots\ldots\ldots\ldots$

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$$\boxed{I_{\mathrm{H}} = \rho a \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r \, \mathrm{d}\theta}$$

Because for $x^2 + y^2 = r^2$: limits of r: $r = 0$ to $r = a$

limits of θ: $\theta = 0$ to $\theta = 2\pi$

$$\begin{aligned} I_{\mathrm{H}} &= \int_R \int \rho r^2 \frac{a}{\sqrt{a^2 - r^2}} r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \rho a \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r \, \mathrm{d}\theta \end{aligned}$$

First we have to evaluate

$$I_r = \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r$$

If we substitute $u = a^2 - r^2$ then the integral is evaluated as

$$I_r = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_r = \frac{2a^3}{3}}$$

Because

When $u = a^2 - r^2$ then $\mathrm{d}u = -2r \, \mathrm{d}r$ so that $r^2 = a^2 - u$ and $r \, \mathrm{d}r = -\frac{\mathrm{d}u}{2}$. Therefore

$$\begin{aligned} I_r &= \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r = \int_{r=0}^a \frac{r^2}{\sqrt{a^2 - r^2}} r \, \mathrm{d}r \\ &= -\int_{u=a^2}^0 \frac{a^2 - u}{\sqrt{u}} \frac{\mathrm{d}u}{2} \\ &= -\frac{a^2}{2} \int_{u=a^2}^0 u^{-1/2} \, \mathrm{d}u + \frac{1}{2} \int_{u=a^2}^0 u^{1/2} \, \mathrm{d}u \\ &= -\frac{a^2}{2} \left[2u^{1/2} \right]_{u=a^2}^0 + \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_{u=a^2}^0 \\ &= a^3 - \frac{a^3}{3} \\ &= \frac{2a^3}{3} \end{aligned}$$

Now, to complete I_{H} we have

$$\begin{aligned} I_{\mathrm{H}} &= \rho a \int_0^{2\pi} \frac{2a^3}{3} \, \mathrm{d}\theta \\ &= \ldots\ldots\ldots\ldots \end{aligned}$$

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$$\boxed{I_H = \frac{4\pi\rho a^4}{3}}$$

Because

$$I_H = \rho a \int_0^{2\pi} \frac{2a^3}{3} d\theta = \frac{2a^4\rho}{3}\Big[\theta\Big]_0^{2\pi} = \frac{4\pi a^4\rho}{3}$$

Therefore, the moment of inertia for the complete spherical shell is

$$I_s = \frac{8\pi a^4\rho}{3}$$

The total mass of the shell $M = 4\pi a^2\rho \quad \therefore\ I = \dfrac{2Ma^2}{3}$

Now let us turn our attention towards *volume integrals* and in preparation review systems of space coordinates.

Space coordinate systems

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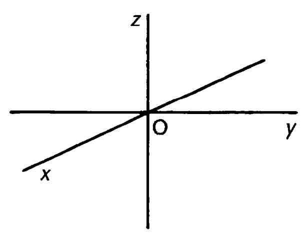
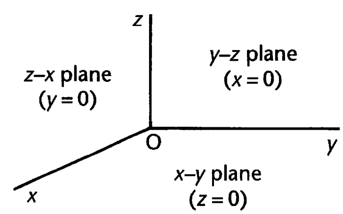
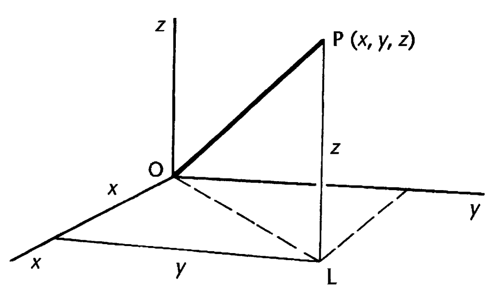
1 *Cartesian coordinates* (x, y, z) – referred to three coordinate axes OX, OY, OZ at right angles to each other. These are arranged in a *right-handed* manner, i.e. turning from OX to OY gives a right-handed screw action in the positive direction of OZ.





The three coordinate planes, $x = 0$, $y = 0$, $z = 0$, divide the space into eight sections called *octants*. The section containing $x \geq 0$, $y \geq 0$, $z \geq 0$ is called the *first octant*.



For a point P (x, y, z)

$$OL^2 = x^2 + y^2$$

$$OP^2 = x^2 + y^2 + z^2$$

Note that this is Pythagoras' theorem in three dimensions.

We are all familiar with this system of coordinates.

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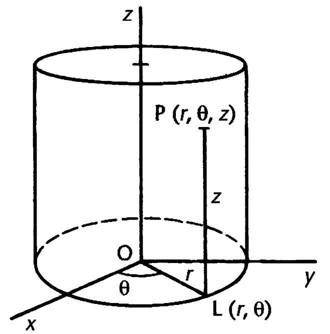
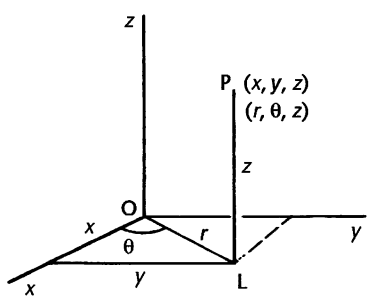
2 *Cylindrical coordinates* (r, θ, z) are useful where an axis of symmetry occurs.



Any point P is considered as having a position on a cylinder. If L is the projection of P on the x–y plane, then (r, θ) are the usual polar coordinates of L. The cylindrical coordinates of P then merely require the addition of the z-coordinate.

$r \geq 0$

Relationship between Cartesian and cylindrical coordinates



If we consider a combined figure, we can easily relate the two systems. Expressing each of the following in terms of the alternative system,

$x = \ldots\ldots\ldots\ldots$	$r = \ldots\ldots\ldots\ldots$
$y = \ldots\ldots\ldots\ldots$	$\theta = \ldots\ldots\ldots\ldots$
$z = \ldots\ldots\ldots\ldots$	$z = \ldots\ldots\ldots\ldots$

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$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\theta$	$\theta = \arctan(y/x)$
$z = z$	$z = z$

So, in cylindrical coordinates, the surface defined by

(1) $r = 5$ is $\ldots\ldots\ldots\ldots$

(2) $\theta = \pi/6$ is $\ldots\ldots\ldots\ldots$

(3) $z = 4$ is $\ldots\ldots\ldots\ldots$

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(1) $r = 5$ is a right cylinder, radius 5, with OZ as axis.
(2) $\theta = \pi/6$ is a plane through OZ, making an angle $\pi/6$ with OX.
(3) $z = 4$ is a plane parallel to the x–y plane cutting OZ at 4 units above the origin.

So position P (2, 3, 4) in Cartesian coordinates
= .... in cylindrical coordinates
and position Q (2·5, $\pi/3$, 6) in cylindrical coordinates
= .... in Cartesian coordinates.

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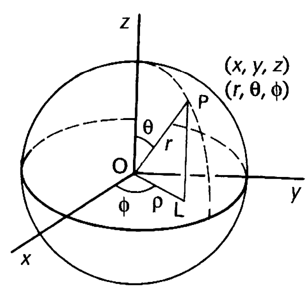
P (2, 3, 4) = ($\sqrt{13}$, 0·983, 4) in cylindrical coordinates
Q (2·5, $\pi/3$, 6) = (1·25, 2·165, 6) in Cartesian coordinates.

3 *Spherical coordinates* (r, θ, ϕ) are appropriate where a centre of symmetry occurs. The position of a point is considered as being a point on a sphere.



r is the distance of P from the origin and is always taken as positive.

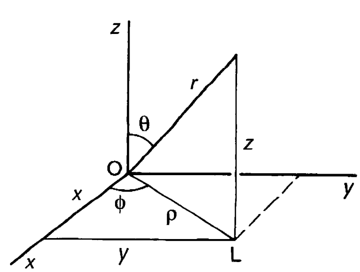
L is the projection of P on the x–y plane

θ is the angle between OP and the positive OZ axis

ϕ is the angle between OL and the OX axis.

Note that (a) ϕ may be regarded as the longitude of P from OX
(b) θ may be regarded as the complement of the latitude of P.

Relationship between Cartesian and spherical coordinates



The combined figure shows the connection between the two systems, so

$x = \ldots\ldots\ldots\ldots$ $r = \ldots\ldots\ldots\ldots$
$y = \ldots\ldots\ldots\ldots$ $\theta = \ldots\ldots\ldots\ldots$
$z = \ldots\ldots\ldots\ldots$ $\phi = \ldots\ldots\ldots\ldots$

28

$$\boxed{\begin{array}{ll} x = r\sin\theta\cos\phi & r = \sqrt{x^2+y^2+z^2} \\ y = r\sin\theta\sin\phi & \theta = \arccos(z/r) \\ z = r\cos\theta & \phi = \arctan(y/x) \end{array}}$$

For the spherical coordinates of any point in space

$r \geq 0$; $0 \leq \theta \leq \pi$; $0 \leq \phi \leq 2\pi$

So, converting Cartesian coordinates (2, 3, 4) to spherical coordinates gives ....

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$$\boxed{\text{P}\,(r,\ \theta,\ \phi) = (5{\cdot}385,\ 0{\cdot}734,\ 0{\cdot}983)}$$

Because

$$x = 2,\ y = 3,\ z = 4$$

$$\therefore\ r = \sqrt{x^2+y^2+z^2} = \sqrt{4+9+16} = \sqrt{29} = 5{\cdot}385$$

$$\theta = \arccos(z/r) = \arccos(4/\sqrt{29}) = 0{\cdot}734$$

$$\phi = \arctan(y/x) = \arctan 1{\cdot}5 = 0{\cdot}983$$

And, in reverse, spherical coordinates $(5, \pi/4, \pi/3)$ transform into Cartesian coordinates ....

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$$\boxed{\text{P}\,(x,\ y,\ z) = (1{\cdot}768,\ 3{\cdot}061,\ 3{\cdot}536)}$$

Because

$$\begin{array}{llll} x = r\sin\theta\cos\phi & = 5\sin\dfrac{\pi}{4}\cos\dfrac{\pi}{3} & = 5(0{\cdot}707)(0{\cdot}5) & = 1{\cdot}768 \\ y = r\sin\theta\sin\phi & = 5\sin\dfrac{\pi}{4}\sin\dfrac{\pi}{3} & = 5(0{\cdot}707)(0{\cdot}866) & = 3{\cdot}061 \\ z = r\cos\theta & = 5\cos\dfrac{\pi}{4} & = 5(0{\cdot}707) & = 3{\cdot}536. \end{array}$$

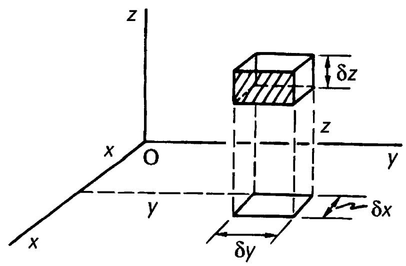
One of the main uses of cylindrical and spherical coordinates occurs in integrals dealing with volumes of solids. In preparation for this, let us consider the next important section of the work.

So move on

31

Element of volume in space in the three coordinate systems

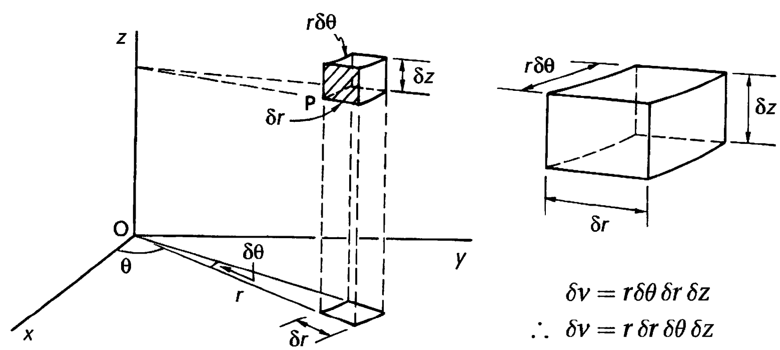
1 *Cartesian coordinates*



We have already used this many times.

$\delta v = \delta x\,\delta y\,\delta z$

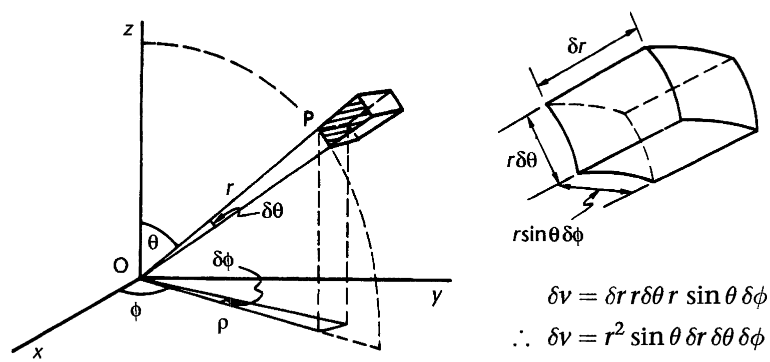
2 *Cylindrical coordinates*



$\delta v = r\delta\theta\,\delta r\,\delta z$

$\therefore\ \delta v = r\,\delta r\,\delta\theta\,\delta z$

3 *Spherical coordinates*



$\delta v = \delta r\,r\delta\theta\,r\,\sin\theta\,\delta\phi$

$\therefore\ \delta v = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$

It is important to make a note of these results, since they are required when we change the variables in various types of integrals. We shall meet them again before long, so be sure of them now.

Volume integrals

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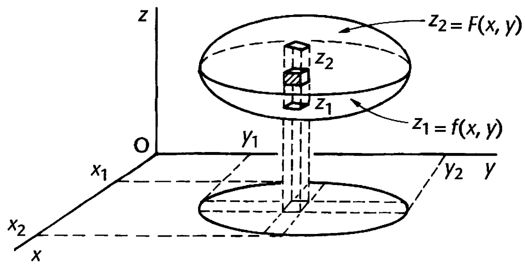


A solid is enclosed by a lower surface $z_1 = f(x, y)$ and an upper surface $z_2 = F(x, y)$.

Then, in general, using Cartesian coordinates, the element of volume is $\delta v = \delta x\,\delta y\,\delta z$.

The approximate value of the total volume V is then found

(a) by summing δv from $z = z_1$ to $z = z_2$ to obtain the volume of the column

(b) by summing all such columns from $y = y_1$ to $y = y_2$ to obtain the volume of the slice

(c) by summing all such slices from $x = x_1$ to $x = x_2$ to obtain the total volume V.

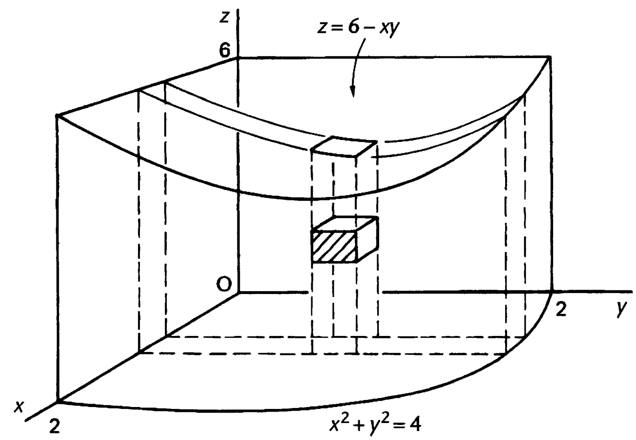
Then, when $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$, the summation becomes an integral

$$V = \int_{x=x_1}^{x=x_2}\int_{y=y_1}^{y=y_2}\int_{z=z_1}^{z=z_2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Example 1

Find the volume of the solid bounded by the planes $z = 0$, $x = 0$, $y = 0$, $x^2 + y^2 = 4$ and $z = 6 - xy$ for $x \geq 0$, $y \geq 0$, $z \geq 0$.

First sketch the figure, so that we can see what we are doing. Take your time over it.



$$\delta v = \delta x\,\delta y\,\delta z$$

$$\text{Volume of column} \approx \sum_{z=0}^{z=6-xy} \delta x\,\delta y\,\delta z$$

$$\text{Volume of slice} \approx \sum_{y=0}^{\sqrt{4-x^2}} \left\{ \sum_{z=0}^{6-xy} \delta x\,\delta y\,\delta z \right\}$$

$$\text{Total volume} \approx \sum_{x=0}^{2} \sum_{y=0}^{\sqrt{4-x^2}} \sum_{z=0}^{6-xy} \delta x\,\delta y\,\delta z$$

If $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$, then

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{6-xy} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Starting with the innermost integral

$$\int_0^{6-xy} \mathrm{d}z = \Big[z\Big]_0^{6-xy}$$
$$= 6 - xy$$

Then $\displaystyle\int_0^{\sqrt{4-x^2}} (6 - xy)\,\mathrm{d}y = \dots\dots\dots\dots$

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$$\boxed{6\sqrt{4-x^2} - \frac{x}{2}(4-x^2)}$$

Because

$$\int_0^{\sqrt{4-x^2}} (6 - xy)\,dy = \left[6y - \frac{xy^2}{2}\right]_{y=0}^{y=\sqrt{4-x^2}}$$
$$= 6\sqrt{4-x^2} - \frac{x}{2}(4-x^2)$$

Then finally $V = \int_0^2 \left\{6(4-x^2)^{1/2} - 2x + \frac{x^3}{2}\right\} dx$

Now we are faced with $\int (4-x^2)^{1/2}\,dx$. You may remember that this is a standard form $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left\{x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a}\right\}$.

If not, to evaluate $\int_0^2 \sqrt{4-x^2}\,dx$, put $x = 2\sin\theta$ and proceed from there.

Finish off the main integral, so that we have

$V = \ldots\ldots\ldots\ldots$

35

$$\boxed{V = 6\pi - 2 \approx 16{\cdot}8 \text{ cubic units}}$$

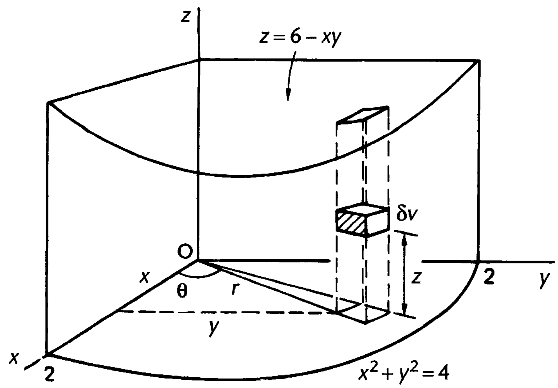
Because we had

$$V = \int_0^2 \left\{6(4-x^2)^{1/2} - 2x + \frac{x^3}{2}\right\} dx$$
$$= 3\left[x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}\right]_0^2 - \left[x^2 - \frac{x^4}{8}\right]_0^2$$
$$= 3\{4\ \arcsin 1 - 4\ \arcsin 0\} - 4 + 2$$
$$= 3\{2\pi\} - 2 = 6\pi - 2$$
$$\approx 16{\cdot}8$$

▶

Alternative method

We could, of course, have used cylindrical coordinates in this problem.



$$\delta v = r\,\delta r\,\delta\theta\,\delta z$$
$$x = r\cos\theta;\ y = r\sin\theta$$
$$\therefore\ z = 6 - xy$$
$$= 6 - r^2\sin\theta\cos\theta$$
$$= 6 - \frac{r^2}{2}\sin 2\theta$$

$$\therefore\ V = \int_{r=0}^{2}\int_{\theta=0}^{\pi/2}\int_{z=0}^{6-(r^2/2)\sin 2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\int_{z=0}^{6-(r^2/2)\sin 2\theta} \mathrm{d}z\,r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \ldots\ldots\ldots\ldots$$

Finish it

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$$\boxed{V = 6\pi - 2 \text{ (as before)}}$$

$$V = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\left(6 - \frac{r^2}{2}\sin 2\theta\right) r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\left(6r - \frac{r^3}{2}\sin 2\theta\right)\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_{0}^{\pi/2}\left[3r^2 - \frac{r^4}{8}\sin 2\theta\right]_{r=0}^{r=2}\mathrm{d}\theta$$
$$= \int_{0}^{\pi/2}(12 - 2\sin 2\theta)\,\mathrm{d}\theta$$
$$= \left[12\theta + \cos 2\theta\right]_{0}^{\pi/2}$$
$$= (6\pi - 1) - 1$$
$$\therefore\ V = 6\pi - 2$$

In this case, the use of cylindrical coordinates facilitates the evaluation.

Let us consider another example.

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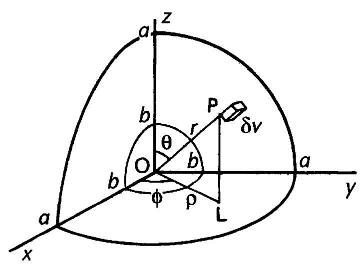
Example 2

To find the moment of inertia and radius of gyration of a thick hollow sphere about a diameter as axis. Outer radius $= a$; inner radius $= b$; density of material $= c$.

It is convenient to deal with one-eighth of the sphere in the first octant.



$\therefore$ Total mass of the solid $M_1 = \dfrac{1}{8}M$

$$M_1 = \frac{1}{8}\cdot\frac{4}{3}\pi(a^3 - b^3)c = \frac{\pi}{6}(a^3 - b^3)c$$

Using spherical coordinates, the element of volume

$$\delta v = \ldots\ldots\ldots\ldots$$

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$$\boxed{\delta v = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi}$$

Also the element of mass $m = c\delta v$

Second moment of mass of the element about OZ

$$\begin{aligned} &= m\rho^2 = m(r\sin\theta)^2 \\ &\qquad = c\,r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi\,r^2\sin^2\theta \\ &\qquad = c\,r^4 \sin^3\theta\,\delta r\,\delta\theta\,\delta\phi \end{aligned}$$

$\therefore$ Total second moment for the solid

$$I_1 \approx \sum_{\phi=0}^{\pi/2}\sum_{\theta=0}^{\pi/2}\sum_{r=b}^{a} c\,r^4\,\delta r\,\sin^3\theta\,\delta\theta\,\delta\phi$$

Then, as usual, if $\delta r \to 0$, $\delta\theta \to 0$, $\delta\phi \to 0$, we finally obtain

$$I_1 = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=b}^{a} c\,r^4\,\mathrm{d}r\,\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

which you can evaluate without any difficulty and obtain

$$I_1 = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_1 = \frac{\pi}{15}(a^5 - b^5)c}$$

Because

$$\begin{aligned}
I_1 &= \int_0^{\pi/2}\int_0^{\pi/2}\left[c\frac{r^5}{5}\right]_b^a \sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\frac{c}{5}(a^5 - b^5)\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{c}{5}(a^5 - b^5)\int_0^{\pi/2}\int_0^{\pi/2}(1 - \cos^2\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{c}{5}(a^5 - b^5)\int_0^{\pi/2}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi/2}\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
&= \frac{c}{5}(a^5 - b^5)\int_0^{\pi/2}\left(1 - \frac{1}{3}\right)\mathrm{d}\phi \\
&= \frac{2c}{15}(a^5 - b^5)\Big[\phi\Big]_0^{\pi/2} = \frac{c\pi}{15}(a^5 - b^5)
\end{aligned}$$

Therefore, the moment of inertia for the whole sphere I is

$$I = 8I_1 \qquad \text{i.e.} \quad I = \frac{8\pi}{15}(a^5 - b^5)c$$

Radius of gyration (k) $\quad Mk^2 = I$

$$\therefore\ k = \ldots\ldots\ldots\ldots$$

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$$\boxed{k = \sqrt{\frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right)}}$$

We had already calculated the total mass $M = \dfrac{4\pi}{3}(a^3 - b^3)c$ and since $I = \dfrac{8\pi}{15}(a^5 - b^5)c$ then

$$\frac{4\pi}{3}(a^3 - b^3)ck^2 = \frac{8\pi}{15}(a^5 - b^5)c$$

$$\therefore\ k^2 = \frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right) \qquad \therefore\ k = \sqrt{\frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right)}$$

We have set the working out in considerable detail, since spherical coordinates may be a new topic. Many of the statements can be streamlined when one is familiar with the system.

Now move on for another example

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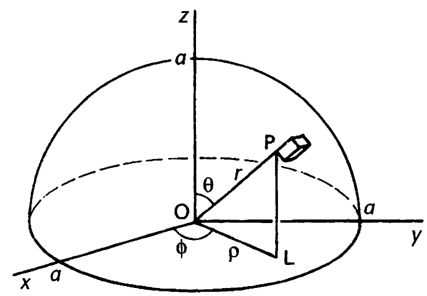
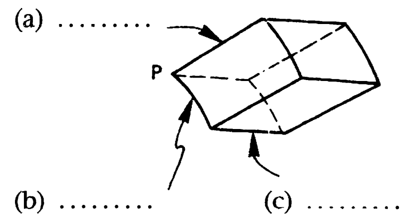
Example 3

Find the total mass of a solid sphere of radius a, enclosed by the surface $x^2 + y^2 + z^2 = a^2$ and having variable density c where $c = 1 + r\,|z|$ and r is the distance of any point from the origin.

This is a case where spherical coordinates can clearly be used with advantage.



In the element of volume, the three dimensions are



42

$$\boxed{\text{(a) } \delta r \quad \text{(b) } r\,\delta\theta \quad \text{(c) } \rho\,\delta\phi = r\sin\theta\,\delta\phi}$$

so that $\delta v = \ldots\ldots\ldots\ldots$

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$$\boxed{\delta v = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi}$$

Then the mass of the element $= c\,\delta v = (1 + r\,|z|)\,\delta v$

and $\qquad z = r\cos\theta$

$$\therefore\ m = c\,\delta v = (1 + r^2\cos\theta)\,r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

Since the density uses $|z| = 1$ we must only consider the region where $\cos\theta \geq 0$ and so we consider the *upper hemisphere* only. The integral for the total mass M_1 is

$$M_1 = \ldots\ldots\ldots\ldots$$

Write out the integral and insert the limits.

44

$$\boxed{M_1 = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a}(1+r^2\cos\theta)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}$$

$$\text{i.e.}\quad M_1 = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\{r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi + r^4\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,d\phi\}$$

$$= \qquad I_1 \qquad + \qquad I_2$$

$$I_1 = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \text{ gives } \ldots\ldots\ldots\ldots$$

Do *not* work it out. You can doubtless recognise what the result would represent.

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The volume of the hemisphere

Because the integral is simply the summation of elements of volume throughout the region of the hemisphere.

Thus, without more ado, $I_1 = \dfrac{2}{3}\pi a^3$.

Now for I_2.

$$I_2 = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a r^4\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$= \ldots\ldots\ldots\ldots$ Evaluate the triple integral.

46

$$\boxed{I_2 = \frac{\pi a^5}{5}}$$

Because

$$I_2 = \int_0^{2\pi}\int_0^{\pi/2}\frac{a^5}{5}\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}\mathrm{d}\phi$$

$$= \frac{a^5}{10}\int_0^{2\pi}1\mathrm{d}\phi$$

$$= \frac{a^5}{10}\Big[\phi\Big]_0^{2\pi} = \frac{\pi a^5}{5}$$

$$\therefore\ I_2 = \frac{\pi a^5}{5}$$

So now finish it off. For the complete sphere

$$M = \ldots\ldots\ldots\ldots$$

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$$\boxed{M = \frac{2\pi a^3}{15}(10 + 3a^2)}$$

Because

$$M_1 = I_1 + I_2 = \frac{2}{3}\pi a^3 + \frac{\pi a^5}{5} = \frac{\pi a^3}{15}(10 + 3a^2)$$

Then, for the whole sphere, $M = 2M_1 = \dfrac{2\pi a^3}{15}(10 + 3a^2)$

Each problem, then, is tackled in much the same way.

(a) Draw a careful sketch diagram, inserting all relevant information.
(b) Decide on the most appropriate coordinate system to use.
(c) Build up the multiple integral and insert correct limits.
(d) Evaluate the integral.

And now we can apply the general guide lines to a final problem.

Example 4

Determine the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = x$, $z = 2$ and $y = 4 - x^2$ in the first quadrant.

First we sketch the diagram.

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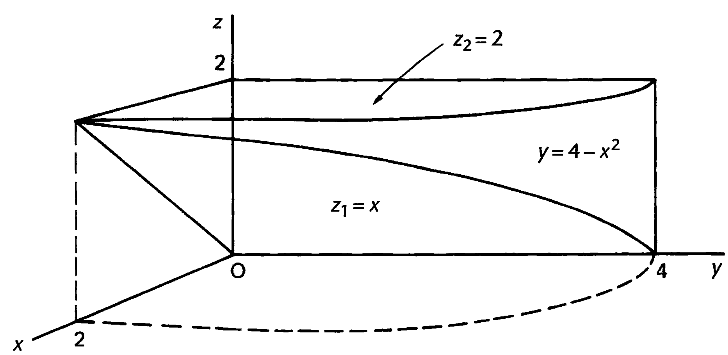

There is no axis of symmetry and no spherical centre. We shall therefore use .... coordinates.

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Cartesian

So off you go on your own. There are no snags.

$V = \ldots\ldots\ldots\ldots$

50

$$\boxed{V = 6\frac{2}{3} \text{ cubic units}}$$

Here is the complete solution.

$$V \approx \sum_{x=0}^{2} \sum_{y=0}^{4-x^2} \sum_{z=x}^{2} \delta x \, \delta y \, \delta z$$

$$\therefore \; V = \int_{x=0}^{2} \int_{y=0}^{4-x^2} \int_{z=x}^{2} \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$

$$= \int_0^2 \int_0^{4-x^2} (2-x) \, \mathrm{d}y \, \mathrm{d}x$$

$$= \int_0^2 \left[2y - xy \right]_{y=0}^{4-x^2} \mathrm{d}x$$

$$= \int_0^2 \{8 - 2x^2 - 4x + x^3\} \mathrm{d}x$$

$$= \left[8x - \frac{2x^3}{3} - 2x^2 + \frac{x^4}{4} \right]_0^2$$

$$= 6\frac{2}{3}$$

And that is it. Now we move to the next section of work

Change of variables in multiple integrals

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In Cartesian coordinates, we use the variables (x, y, z); in cylindrical coordinates, we use the variables (r, θ, z); in spherical coordinates, we use the variables (r, θ, ϕ); and we have established relationships connecting these systems of variables, permitting us to transfer from one system to another. These relationships, you will remember, were obtained geometrically in Frames 23 to 30 of this Programme.

There are occasions, however, when it is expedient to make other transformations beside those we have used and it is worth looking at the problem in a rather more general manner.

This we will now do

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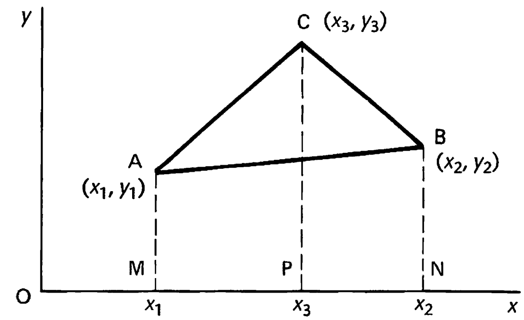
First, however, let us revise a result from an earlier Programme on determinants to find the area of the triangle ABC.



If we arrange the vertices A (x_1, y_1)
B (x_2, y_2)
C (x_3, y_3)

in an anticlockwise manner then

area triangle ABC = trapezium AMPC + trapezium CPNB − trapezium AMNB

$$= \tfrac{1}{2}\{(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)\}$$

$$= \tfrac{1}{2}\{x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 + x_2y_2 + x_2y_3 - x_3y_2 - x_3y_3 - x_2y_1 - x_2y_2 + x_1y_1 + x_1y_2\}$$

$$= \tfrac{1}{2}\{(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

The determinant is positive if the points A, B, C are taken in an anticlockwise manner.

We shall need to use this result in a short while, so keep it in mind.

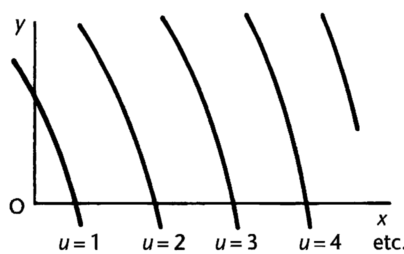
On to the next frame

Curvilinear coordinates

Consider the double integral $\int_R\int \phi(x, y)\mathrm{d}A$ where $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ in Cartesian coordinates. Let u and v be two new independent variables defined by $u = F(x, y)$ and $v = G(x, y)$ where these equations can be simultaneously solved to obtain $x = f(u, v)$ and $y = g(u, v)$. Furthermore, these transformation equations are such that every point (x, y) is mapped to a unique point (u, v) and vice versa. **53**

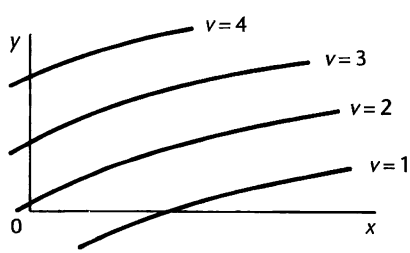
Let us see where this leads us, so on to the next frame

The equation $u = F(x, y)$ will be a family of curves depending on the particular constant value given to u in each case. **54**



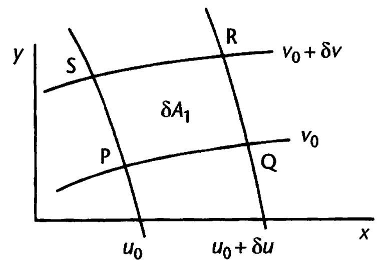
Curves $u = F(x, y)$ for different constant values of u.

Similarly, $v = G(x, y)$ will be a family of curves depending on the particular constant value assigned to v in each case.



Curves $v = G(x, y)$ for different constant values of v.

These two sets of curves will therefore cover the region R and form a network, and to any point P (x_0, y_0) there will be a pair of curves $u = u_0$ (constant) and $v = v_0$ (constant) that intersect at that point.



The u- and v-values relating to any particular point are known as its *curvilinear coordinates* and $x = f(u, v)$ and $y = g(u, v)$ are the *transformation equations* between the two systems.

In the Cartesian coordinates (x, y) system, the element of area $\delta A = \delta x \delta y$ and is the area bounded by the lines $x = x_0$, $x = x_0 + \delta x$, $y = y_0$, and $y = y_0 + \delta y$.

In the new system of *curvilinear coordinates* (u, v) the element of area δA_1 can be taken as that of the figure P, Q, R, S, i.e. the area bounded by the curves $u = u_0$, $u = u_0 + \delta u$, $v = v_0$ and $v = v_0 + \delta v$.

Since δA_1 is small, PQRS may be regarded as a parallelogram

i.e. $\delta A_1 \approx 2 \times$ area of triangle PQS

and this is where we make use of the result previously revised that the area of a triangle ABC with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) can be expressed in determinant form as

Area = ...

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$$\boxed{\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}$$

Before we can apply this, we must find the Cartesian coordinates of P, Q and S in the diagram on page 646 where we omit the subscript $_0$ on the coordinates.

If $x = f(u, v)$, then a small increase δx in x is given by

$$\delta x = \dots\dots\dots\dots$$

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$$\boxed{\delta x = \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v}\delta v}$$

i.e. $\delta x = \dfrac{\partial x}{\partial u}\delta u + \dfrac{\partial x}{\partial v}\delta v$

and, for $y = g(u, v)$

$$\delta y = \dots\dots\dots\dots$$

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$$\boxed{\delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v}$$

Now

(a) P is the point (x, y)

(b) Q corresponds to small changes from P.

$$\delta x = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v \quad \text{and} \quad \delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v$$

But along PQ v is constant. $\therefore\ \delta v = 0$.

$$\therefore\ \delta x = \frac{\partial x}{\partial u}\delta u \quad \text{and} \quad \delta y = \frac{\partial y}{\partial u}\delta u$$

i.e. Q is the point $\left(x + \dfrac{\partial x}{\partial u}\delta u,\ y + \dfrac{\partial y}{\partial u}\delta u\right)$.

(c) Similarly for S, since u is constant along PS $\delta u = 0$ and

$\therefore$ S is the point $\left(x + \dfrac{\partial x}{\partial v}\delta v,\ y + \dfrac{\partial y}{\partial v}\delta v\right)$

So the Cartesian coordinates of P, Q, S are

$$\text{P } (x, y);\quad \text{Q } \left(x + \frac{\partial x}{\partial u}\delta u,\ y + \frac{\partial y}{\partial u}\delta u\right);\quad \text{S } \left(x + \frac{\partial x}{\partial v}\delta v,\ y + \frac{\partial y}{\partial v}\delta v\right)$$

$\therefore$ The determinant for the area PQS is $\dots\dots\dots\dots$

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$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & x+\dfrac{\partial x}{\partial u}\delta u & x+\dfrac{\partial x}{\partial v}\delta v \\ y & y+\dfrac{\partial y}{\partial u}\delta u & y+\dfrac{\partial y}{\partial v}\delta v \end{vmatrix}$$

Subtracting column 1 from columns 2 and 3 gives

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ x & \dfrac{\partial x}{\partial u}\delta u & \dfrac{\partial x}{\partial v}\delta v \\ y & \dfrac{\partial y}{\partial u}\delta u & \dfrac{\partial y}{\partial v}\delta v \end{vmatrix}$$

which simplifies immediately to

....

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$$\text{Area} = \frac{1}{2}\begin{vmatrix} \dfrac{\partial x}{\partial u}\delta u & \dfrac{\partial x}{\partial v}\delta v \\ \dfrac{\partial y}{\partial u}\delta u & \dfrac{\partial y}{\partial v}\delta v \end{vmatrix}$$

Then, taking out the factor δu from the first column and the factor δv from the second column, this becomes

Area = ...

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$$\frac{1}{2}\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\delta u\delta v$$

The area of the approximate parallelogram is twice the area of the triangle.

$$\therefore \text{ Area of parallelogram} = \delta A_1 = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\delta u\,\delta v$$

Expressing this in differentials

$$\mathrm{d}A = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathrm{d}u\,\mathrm{d}v$$

and, for convenience, this is often written

$$\mathrm{d}A = \frac{\partial(x, y)}{\partial(u, v)}\,\mathrm{d}u\,\mathrm{d}v$$

$\dfrac{\partial(x, y)}{\partial(u, v)}$ is called the *Jacobian of the transformation* from the Cartesian coordinates (x, y) to the curvilinear coordinates (u, v).

$$\therefore\; J(u,v) = \frac{\partial(x, y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

So, if the transformation equations are

$$x = u(u+v) \quad \text{and} \quad y = uv^2$$
$$J(u, v) = \dots\dots\dots\dots$$

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$$\boxed{J(u,v) = uv(4u+v)}$$

Because

$$\frac{\partial x}{\partial u} = 2u+v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = v^2 \qquad \frac{\partial y}{\partial v} = 2uv$$

$$\therefore\; J(u,v) = \begin{vmatrix} 2u+v & u \\ v^2 & 2uv \end{vmatrix} = 4u^2v + 2uv^2 - uv^2$$

$$= 4u^2v + uv^2 = uv(4u+v)$$

Next frame

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Sometimes the transformation equations are given the other way round. That is, where u and v are given as expressions in x and y. In such a case $J(u, v)$ can be found using the fact that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\left(\dfrac{\partial(u, v)}{\partial(x, y)}\right)}$$

For example, if the transformation equations are given as $u = x^2 + y^2$ and $v = 2xy$ then

$$J(u, v) = \dots\dots\dots\dots$$

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$$\boxed{J(u,\,v) = \frac{1}{4\sqrt{u^2 - v^2}}}$$

Because

$$\frac{\partial(u,\,v)}{\partial(x,\,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} = 4x^2 - 4y^2$$

and so

$$J(u,\,v) = \frac{\partial(x,\,y)}{\partial(u,\,v} = \frac{1}{\left(\dfrac{\partial(u,\,v)}{\partial(x,\,y)}\right)} = \frac{1}{4(x^2 - y^2)}$$

Now $u - v = x^2 - 2xy + y^2 = (x - y)^2$
and $u + v = x^2 + 2xy + y^2 = (x + y)^2$
and so $x^2 - y^2 = (x - y)(x + y) = \sqrt{u - v}\sqrt{u + v} = \sqrt{u^2 - v^2}$ giving

$$J(u,\,v) = \frac{1}{4\sqrt{u^2 - v^2}}$$

There is one further point to note in this piece of work, so move on

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Note: In the transformation, it is possible for the order of the points P, Q, R, S to be reversed with the result that δA may give a negative result when the determinant is evaluated. To ensure a positive element of area, the result is finally written

$$\mathrm{d}A = \left|\frac{\partial(x,\,y)}{\partial(u,\,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

where the 'modulus' lines indicate the absolute value of the Jacobian.

Therefore, to rewrite the integral $\int_R\int F(x,\,y)\,\mathrm{d}x\,\mathrm{d}y$ in terms of the new variables, u and v, where $x = f(u, v)$ and $y = g(u, v)$, we substitute for x and y in $F(x,\,y)$ and replace $\mathrm{d}x\,\mathrm{d}y$ with $\left|\dfrac{\partial(x,\,y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v$.

The integral then becomes

$$\int_R\int F\{f(u,\,v),\,g(u, v)\} \left|\frac{\partial(x,\,y)}{\partial(u,\,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

Make a note of this result

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Example 1

Express $I = \int_R \int xy^2 \, dx \, dy$ in polar coordinates, making the substitutions $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \; J(r, \theta) = \dots\dots\dots\dots$$

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$$\boxed{J(r, \theta) = r}$$

$$J(r, \theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Then $I = \int_R \int xy^2 \, dx \, dy$ becomes ....

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$$\boxed{I = \int_R \int r^3 \sin^2\theta \cos\theta \, r \, dr \, d\theta}$$

Because $xy^2 = r\cos\theta \, r^2 \sin^2\theta = r^3 \sin^2\theta \cos\theta$

$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr \, d\theta = r \, dr \, d\theta$$

$$\therefore \; I = \int_R \int r^3 \sin^2\theta \cos\theta \, r \, dr \, d\theta = \int_R \int r^4 \sin^2\theta \cos\theta \, dr \, d\theta$$

Now this one.

Example 2

Express $I = \int_R \int (x^2 + y^2) dx \, dy$ in terms of u and v, given that $x = u^2 - v^2$ and $y = 2uv$.

First of all, the expression for $\dfrac{\partial(x, y)}{\partial(u, v)}$ gives ....

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$$\boxed{4(u^2+v^2)}$$

Because

$$x = u^2 - v^2 \quad \therefore \quad \frac{\partial x}{\partial u} = 2u \quad \frac{\partial x}{\partial v} = -2v$$

$$y = 2uv \quad \therefore \quad \frac{\partial y}{\partial u} = 2v \quad \frac{\partial y}{\partial v} = 2u$$

$$\therefore \quad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4(u^2+v^2)$$

Also $x^2 + y^2 = (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2$
$= u^4 + 2u^2v^2 + v^4 = (u^2+v^2)^2$

Then $I = \int\int_R (x^2+y^2)\,dx\,dy$ becomes $I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = 4\int_R\int (u^2+v^2)^3 \,du\,dv}$$

One more.

Example 3

By substituting $x = 2uv$ and $y = u(1-v)$ where $u > 0$ and $v > 0$, express the integral $I = \int_R\int x^2y\,dx\,dy$ in terms of u and v.

Complete it: there are no snags. $I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = 8\int_R\int u^4v^2(1-v)\,\mathrm{d}u\,\mathrm{d}v}$$

Working:

$$x = 2uv \qquad \therefore \quad \frac{\partial x}{\partial u} = 2v \qquad \frac{\partial x}{\partial v} = 2u$$

$$y = u - uv \qquad \frac{\partial y}{\partial u} = 1 - v \qquad \frac{\partial y}{\partial v} = -u$$

$$\therefore \; J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2v & 1-v \\ 2u & -u \end{vmatrix}$$

$$= 2u\begin{vmatrix} v & 1-v \\ 1 & -1 \end{vmatrix} = 2u\begin{vmatrix} v & 1 \\ 1 & 0 \end{vmatrix} = -2u$$

$$\therefore \; \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 2u$$

$$x^2y = 4u^2v^2(u - uv) = 4u^3v^2(1-v)$$

$$\therefore \; I = \int_R\int 4u^3v^2(1-v)\,2u\,\mathrm{d}u\,\mathrm{d}v$$

$$I = 8\int_R\int u^4v^2(1-v)\,\mathrm{d}u\,\mathrm{d}v$$

Transformation in three dimensions

If we extend the previous results to convert variables (x, y, z) to (u, v, w), we proceed in just the same way.

If $x = f(u, v, w)$; $y = g(u, v, w)$; $z = h(u,v,w)$

Then $$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\[2ex] \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

and the element of volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ becomes

$$\mathrm{d}V = |J(u,v,w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Also $\int\int\int F(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is transformed into

$$\int\int\int G(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Now for an example, so move on

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Example 4

To transform a triple integral $I = \iiint F(x, y, z)\,dx\,dy\,dz$ in Cartesian coordinates to spherical coordinates by the transformation equations

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

First we need the partial derivatives, from which to build up the Jacobian.

These are ....

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$$\boxed{\begin{array}{lll} \dfrac{\partial x}{\partial r} = \sin\theta\cos\phi & \dfrac{\partial y}{\partial r} = \sin\theta\sin\phi & \dfrac{\partial z}{\partial r} = \cos\theta \\ \dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi & \dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi & \dfrac{\partial z}{\partial \theta} = -r\sin\theta \\ \dfrac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi & \dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi & \dfrac{\partial z}{\partial \phi} = 0 \end{array}}$$

$$\therefore\ J(r,\theta,\phi) = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\phi & r\cos\theta\sin\phi \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$+ r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{r^2\sin\theta}$$

Because

$$J(r,\,\theta,\,\phi) = r^2\cos^2\theta\sin\theta \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix}$$

$$+ r^2\sin^3\theta \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix}$$

$$= (r^2\sin^3\theta + r^2\sin\theta\cos^2\theta) \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix}$$

$$= r^2\sin\theta(\sin^2\theta + \cos^2\theta)(\cos^2\phi + \sin^2\phi) = r^2\sin\theta$$

$$\therefore\ I = \iiint G(u,v,w)r^2\sin\theta\,dr\,d\theta\,d\phi$$

which agrees, of course, with the result we had previously obtained by a geometric consideration.

▶

And that is about it. Check carefully down the **Revision summary** and the **Can You?** checklist that now follow, before working through the **Test exercise**. The **Further problems** give additional practice.

Revision summary 15

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1 *Surface integrals*

$$I = \int_R f(x, y)\,\mathrm{d}a = \int_R \int f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

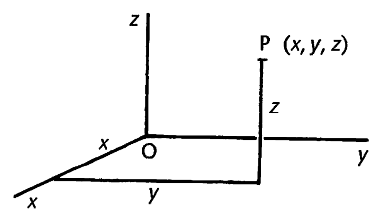
2 *Surface in space*

$$I = \int_S \phi(x,y,z)\,\mathrm{d}S = \int_R \int \phi(x,y,z)\sec\gamma\,\mathrm{d}x\,\mathrm{d}y \qquad (\gamma < \pi/2)$$

$$= \int_R \int \phi(x,y,z)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$$

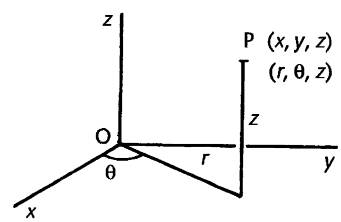
3 *Space coordinate systems*

(a) *Cartesian coordinates* (x, y, z)



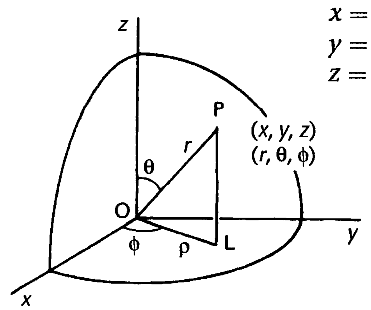
First octant:
$x \geq 0;\ \ y \geq 0;\ \ z \geq 0$

(b) *Cylindrical coordinates* (r, θ, z) $\quad r \geq 0$



$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \theta = \arctan(y/x)$$
$$z = z \qquad z = z$$

(c) *Spherical coordinates* (r, θ, ϕ) $\quad r \geq 0$

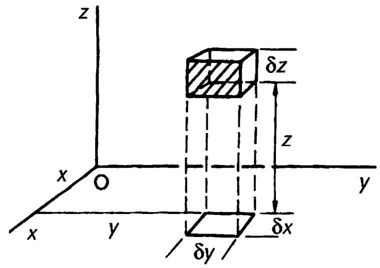


$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$
$$y = r\sin\theta\sin\phi \qquad \theta = \arccos(z/r)$$
$$z = r\cos\theta \qquad \phi = \arctan(y/x)$$

▶

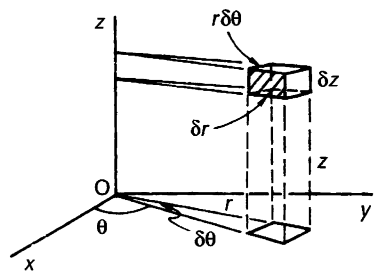
4 *Elements of volume*

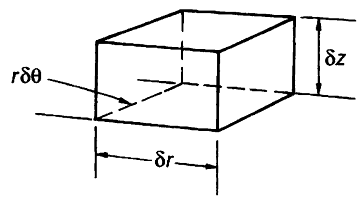
(a) *Cartesian coordinates*



$$\delta v = \delta x\,\delta y\,\delta z$$

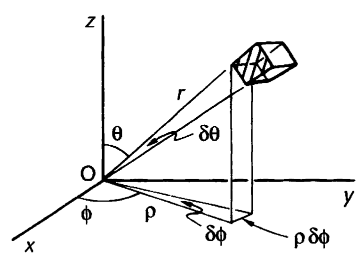
(b) *Cylindrical coordinates* $r \geq 0$

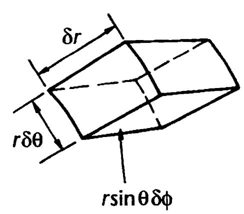




$$\delta v = r\,\delta r\,\delta\theta\,\delta z$$

(c) *Spherical coordinates*





$$\delta v = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

5 *Volume integrals*

$$V = \int\!\!\int\!\!\int \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$I = \int\!\!\int\!\!\int f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

▶

6 *Change of variables* in multiple integrals

(a) *Double integrals* $x = f(u, v); \quad y = g(u, v)$

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv; \qquad J(u,v) = \frac{\partial(x, y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$I = \int_R\int F(x, y)\,dx\,dy = \int_R\int F\{f(u, v), g(u, v)\}\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

(b) *Triple integrals* $x = f(u, v, w); y = g(u, v, w); z = h(u, v, w)$

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$\text{Then } I = \int\int\int F(x, y, z)\,dx\,dy\,dz$$

$$= \int\int\int G(u, v, w)\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw$$

Can You?

Checklist 15

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5, how confident are you that you can: Frames

- Evaluate double integrals and surface integrals? 1 to 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Relate three-dimensional Cartesian coordinates to cylindrical and spherical polar forms? 23 to 31

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate volume integrals in Cartesian coordinates and in cylindrical and spherical polar coordinates? 32 to 50

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the Jacobian to convert integrals given in Cartesian coordinates into general curvilinear coordinates in two and three dimensions? 51 to 73

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 15

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1 Determine the area of the surface $z = \sqrt{x^2 + y^2}$ over the region bounded by $x^2 + y^2 = 4$.

2 Evaluate the surface integral $I = \int_S \phi \, dS$ where $\phi = \dfrac{1}{\sqrt{x^2 + y^2}}$ over the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

3 (a) Transform the Cartesian coordinates
   (1) (4, 2, 3) to cylindrical coordinates (r, θ, z)
   (2) (3, 1, 5) to spherical coordinates (r, θ, ϕ).
   (b) Express in Cartesian coordinates (x, y, z)
   (1) the cylindrical coordinates $(5, \pi/4, 3)$
   (2) the spherical coordinates $(4, \pi/6, 2)$.

4 Determine the volume of the solid bounded by the plane $z = 0$ and the surfaces $x^2 + y^2 = 4$ and $z = x^2 + y^2 + 1$.

5 Determine the total mass of a solid hemisphere bounded by the plane $z = 0$ and the surface $x^2 + y^2 + z^2 = a^2$ $(z \geq 0)$ if the density at any point is given by $\rho = 1 - z$ $(z < a)$.

6 (a) Express the integral $I = \int_R \int (x - y) \, dx \, dy$ in terms of u and v, where $x = u(1 + v)$ and $y = u - v$.
   (b) Express the triple integral $I = \iiint \left(\dfrac{x + z}{y}\right) dx \, dy \, dz$ in terms of u, v, w using the transformation equations
   $x = u + v + w$; $y = v^2 w$; $z = u - w$.

Further problems 15

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1 Evaluate the surface integral $I = \int_S (x^2 + y^2) \, dS$ over the surface of the cone $z^2 = 4(x^2 + y^2)$ between $z = 0$ and $z = 4$.

2 Find the position of the centre of gravity of that part of a thin spherical shell $x^2 + y^2 + z^2 = a^2$ which exists in the first octant.

3 Determine the surface area of the plane $6x + 3y + 4z = 60$ cut off by $x = 0$, $y = 0$, $x = 5$, $y = 8$.

4 Find the surface area of the plane $3x + 2y + 3z = 12$ cut off by the planes $x = 0$, $y = 0$, and the cylinder $x^2 + y^2 = 16$ for $x \geq 0$, $y \geq 0$.

5 Determine the area of the paraboloid $z = 2(x^2 + y^2)$ cut off by the cone $z = \sqrt{x^2 + y^2}$.

6 Find the area of the cone $z^2 = 4(x^2 + y^2)$ which is inside the paraboloid $z = 2(x^2 + y^2)$.

7 Cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ intersect. Determine the total external surface area of the common portion.

8 Determine the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by the cylinder $x^2 + y^2 = ax$.

9 A cylinder of radius b, with the z-axis as its axis of symmetry, is removed from a sphere of radius a, $a > b$, with centre at the origin. Calculate the total curved surface area of the ring so formed, including the inner cylindrical surface.

10 Find the volume enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 5 - x$.

11 Determine the volume of the solid bounded by the surfaces $y = x^2$, $x = y^2$, $z = 2$ and $x + y + z = 4$.

12 Find the volume of the solid bounded by the plane $z = 0$, the cylinder $x^2 + y^2 = a^2$ and the surface $z = x^2 + y^2$.

13 A solid is bounded by the planes $x = 0$, $y = 0$, $z = 2$, $z = x$ and the surface $x^2 + y^2 = 4$. Determine the volume of the solid.

14 Find the position of the centre of gravity of the part of the solid sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

15 A solid is bounded by the cone $z = 2\sqrt{x^2 + y^2}$, $z \geq 0$, and the sphere $x^2 + y^2 + (z - a)^2 = 2a^2$. Determine the volume of the solid so formed.

16 Determine the volume enclosed by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

17 Find the volume of the solid in the first octant bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = x + y$ and the surface $x^2 + y^2 = a^2$.

18 Express the integral $\displaystyle\iint (x^2 + y^2)\,dx\,dy$ in terms of u and v, using the transformations $u = x + y$, $v = x - y$.

19 Determine an expression for the element of volume $dx\,dy\,dz$ in terms of u, v, w using the transformations $x = u(1 - v)$, $y = uv$, $z = uvw$.

20 A solid sphere of radius a has variable density c at any point (x, y, z) given by $c = k(a - z)$ where k is a constant. Determine the position of the centre of gravity of the sphere.

21 Calculate $\displaystyle\iint x^2y^2\,dx\,dy$ over the triangular region in the x–y plane with vertices (0, 0), (1, 1), (1, 2).

▶

22 Evaluate the integral $I = \int_0^2 \int_{\sqrt{y(2-y)}}^{\sqrt{4-y^2}} \frac{y}{x^2+y^2}\,dx\,dy$ by transforming to polar coordinates.

23 Evaluate $I = \int_0^1 \int_0^y \frac{xy^2}{\sqrt{x^2+y^2}}\,dx\,dy$.

24 Find the volume bounded by the cylinder $x^2 + y^2 = a^2$, the plane $z = 0$ and the surface $z = x^2 + y^2$. Convert to polar coordinates and show that $V = \frac{\pi a^4}{2}$.

25 By changing the order of integration in the integral

$$I = \int_0^a \int_x^a \frac{y^2\,dy\,dx}{\sqrt{x^2+y^2}}$$

show that $I = \frac{1}{3}a^3 \ln(1 + \sqrt{2})$.